HiggsTools Journal Club XI

9 February 2016

Introduction to `SecDec`

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An Apology

(First Half) Apology to Theorists:

Talk will be slow, basic and will skip a lot of very important details and steps

(Second Half) Apology to Experimentalists:

Talk will get technical

Don't worry at the end I'll introduce a tool that handles all the book-keeping.

Content

Part 1

- From Cross-sections to Amplitudes
- Feynman Rules
- Loops ↔ Integrals
- Dimensional Regularisation

Part 2

- Feynman Parameters
- Graph Polynomials
- Sector Decomposition

Part 3

- `SecDec` Demo (Implements all of the above)

Schematics

Total CS **Order**

$$\sigma_F = \sigma_F^{(0)} + \sigma_F^{(1)} + \ldots$$

Final State

$$\sigma_F^{(0)} = \sum_{i,j} \int_0^1 \mathrm{d}x_i \int_0^1 \mathrm{d}x_j f_i(x_i) f_j(x_j) \int_m \mathrm{d}\hat{\sigma}_m^{(0)}$$

$$\sigma_F^{(1)} = \sum_{i,j} \int_0^1 \mathrm{d}x_i \int_0^1 \mathrm{d}x_j f_i(x_i) f_j(x_j) \left[\int_m \mathrm{d}\hat{\sigma}_m^{(1)} + \int_{m+1} \mathrm{d}\hat{\sigma}_{m+1}^{(0)} \right]$$

Schematics

Total CS **Order**

$$\sigma_F = \sigma_F^{(0)} + \sigma_F^{(1)} + \ldots$$

Final State

Phase Space Integral

PDFs

(Differential) Partonic CS

$$\sigma_F^{(0)} = \sum_{i,j} \int_0^1 \mathrm{d}x_i \int_0^1 \mathrm{d}x_j f_i(x_i) f_j(x_j) \int_m \mathrm{d}\hat{\sigma}_m^{(0)}$$

 # legs

$$\sigma_F^{(1)} = \sum_{i,j} \int_0^1 \mathrm{d}x_i \int_0^1 \mathrm{d}x_j f_i(x_i) f_j(x_j) \left[\int_m \mathrm{d}\hat{\sigma}_m^{(1)} + \int_{m+1} \mathrm{d}\hat{\sigma}_{m+1}^{(0)} \right]$$

Schematics

Total CS **Order**

$$\sigma_F = \sigma_F^{(0)} + \sigma_F^{(1)} + \ldots$$

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PDFs

(Differential) Partonic CS

$$\sigma_F^{(0)} = \sum_{i,j} \int_0^1 \mathrm{d}x_i \int_0^1 \mathrm{d}x_j f_i(x_i) f_j(x_j) \int_m \mathrm{d}\hat{\sigma}_m^{(0)}$$

 # legs

Higher Order **More Legs**

$$\sigma_F^{(1)} = \sum_{i,j} \int_0^1 \mathrm{d}x_i \int_0^1 \mathrm{d}x_j f_i(x_i) f_j(x_j) \left[\int_m \mathrm{d}\hat{\sigma}_m^{(1)} + \int_{m+1} \mathrm{d}\hat{\sigma}_{m+1}^{(0)} \right]$$

“Virtuals” **“Reals”**

Schematics (II)

(Differential)
Partonic CS

Phase Space Measure

Average/Sum
(Initial/Final)
Spin & Colour

$$\mathrm{d}\hat{\sigma}_m^{(0)} = \mathrm{d}\Phi_m \langle \mathcal{M}_m^{(0)} \mathcal{M}_m^{(0)\dagger} \rangle$$

$$\mathrm{d}\hat{\sigma}_{m+1}^{(0)} = \mathrm{d}\Phi_{m+1} \langle \mathcal{M}_{m+1}^{(0)} \mathcal{M}_{m+1}^{(0)\dagger} \rangle$$

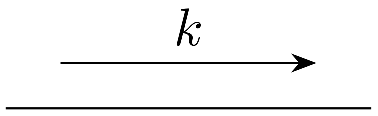
$$\mathrm{d}\hat{\sigma}_m^{(1)} = \mathrm{d}\Phi_m \langle \mathcal{M}_m^{(1)} \mathcal{M}_m^{(0)\dagger} + \mathcal{M}_m^{(1)\dagger} \mathcal{M}_m^{(0)} \rangle$$

Amplitude

Feynman Rules

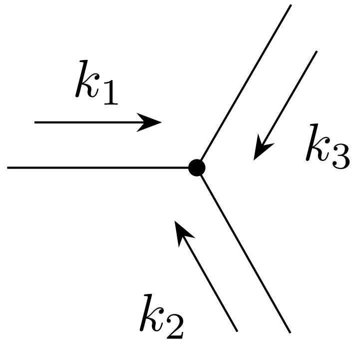
Feynman rules allow us to compute an **amplitude**, $\mathcal{M}$, as an expansion in the coupling, g:

Propagator



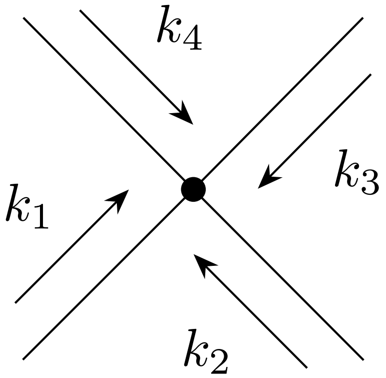
$$\int_{-\infty}^{\infty} \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{(k^2 - m^2 + i\delta)}$$

Vertex (3-point)



$$g\delta^{(4)}(k_1 + k_2 + k_3)$$

Vertex (4-point)

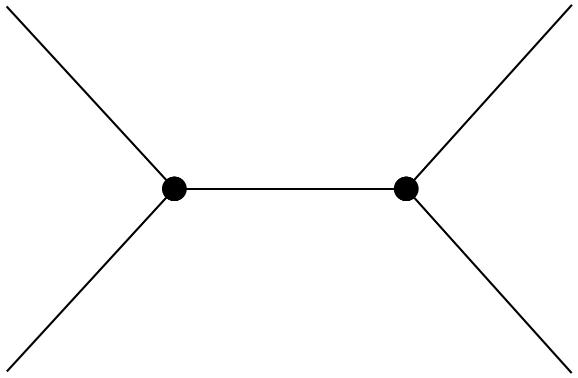


$$g^2\delta^{(4)}(k_1 + k_2 + k_3 + k_4)$$

Corresponds to summing over intermediate states

Propagators increment # integrations, Vertices decrement

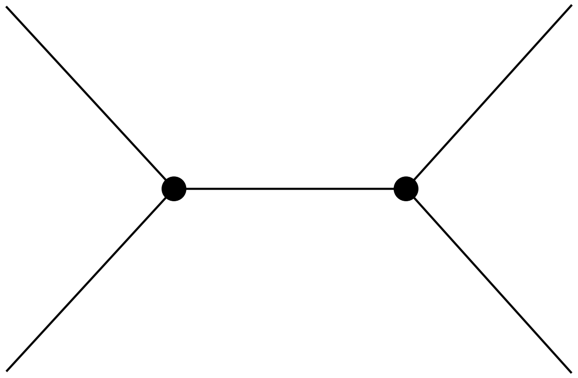
Feynman diagram: \`Glue' these pictures together and \`factor out' a delta function for overall momentum conservation



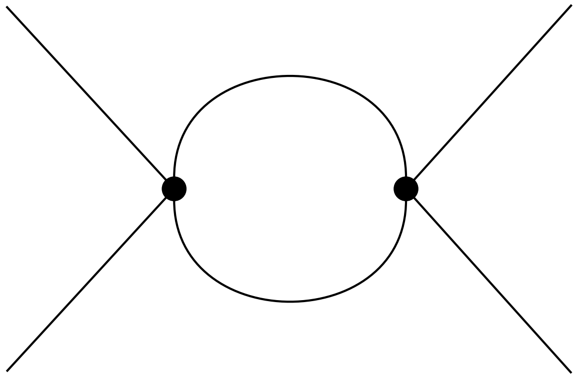
Loops & Integrals

I = # internal lines, V = # vertices

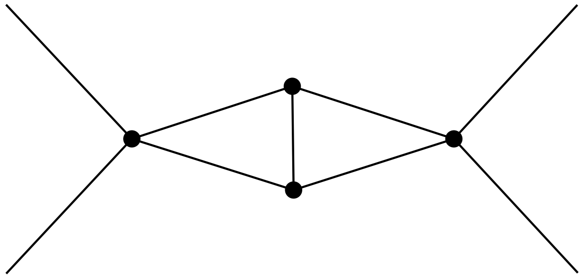
Count the number of unconstrained momenta and call this number L



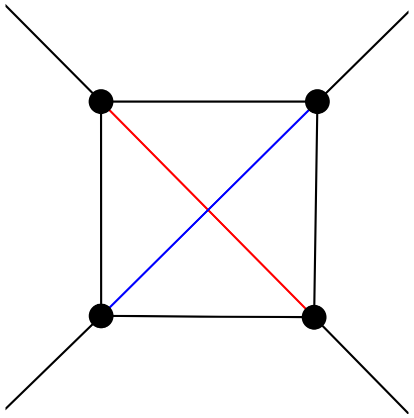
I = 1
V = 2
L = 0



I = 2
V = 2
L = 1



I = 5
V = 4
L = 2



I = 6
V = 4
L =

Overall momentum conservation

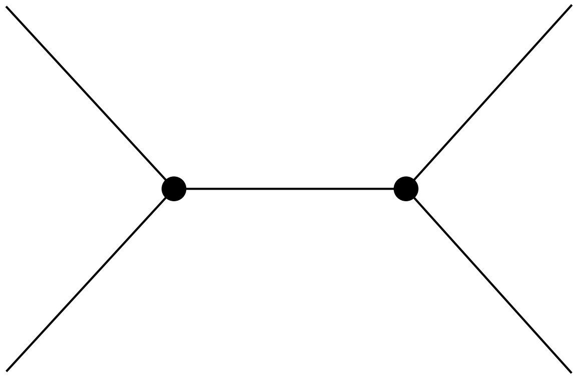
Generally: $L = I - (V - 1)$, We define, L to be **# loops**

Loops ≡ # Unconstrained Momenta ↔ # of Integrations

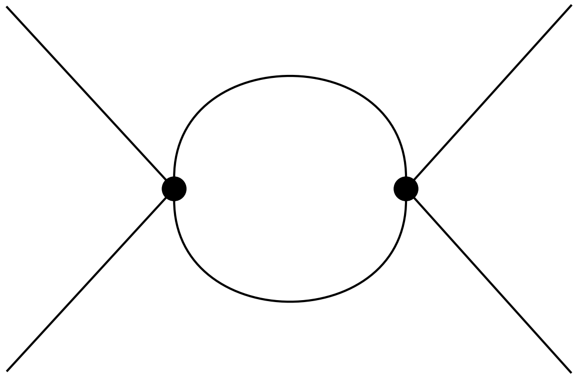
Loops & Integrals

I = # internal lines, V = # vertices

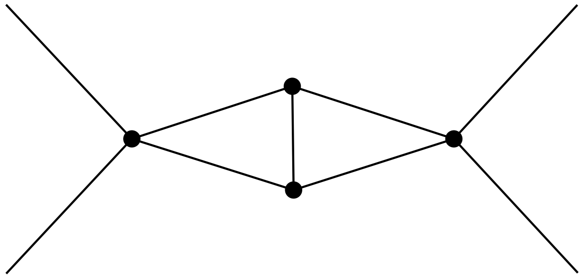
Count the number of unconstrained momenta and call this number L



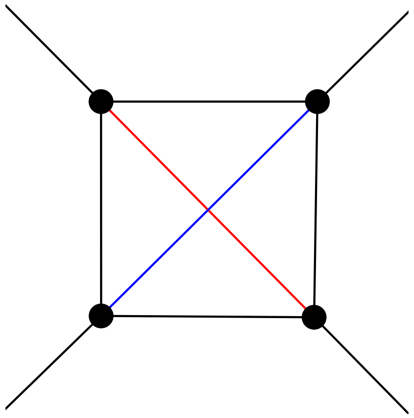
I = 1
V = 2
L = 0



I = 2
V = 2
L = 1



I = 5
V = 4
L = 2



I = 6
V = 4
L = 3

Overall momentum conservation

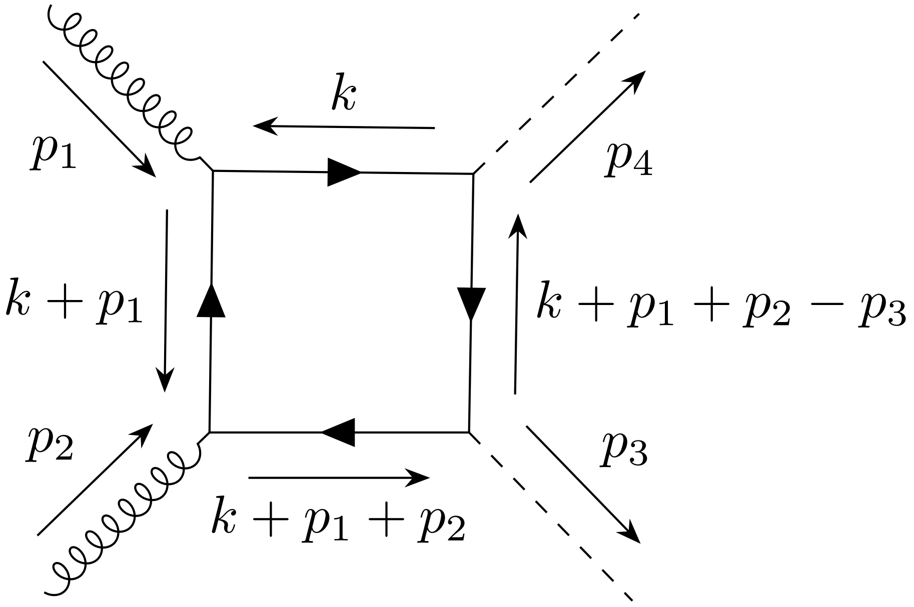
Generally: $L = I - (V - 1)$, We define, L to be **# loops**

Loops ≡ # Unconstrained Momenta ↔ # of Integrations

Constructing Integrals

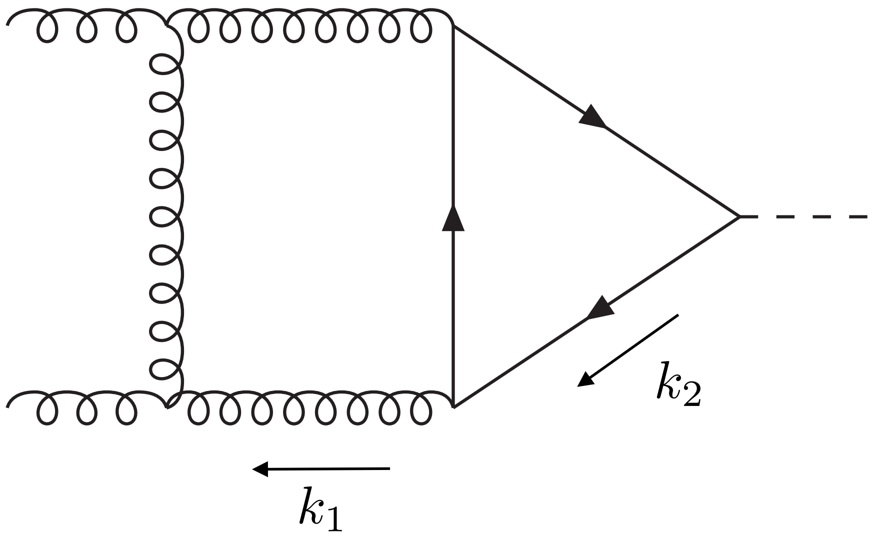
Finding all the integrals ⇒ compute the diagram

Nevertheless, can see the denominator of integrals immediately:



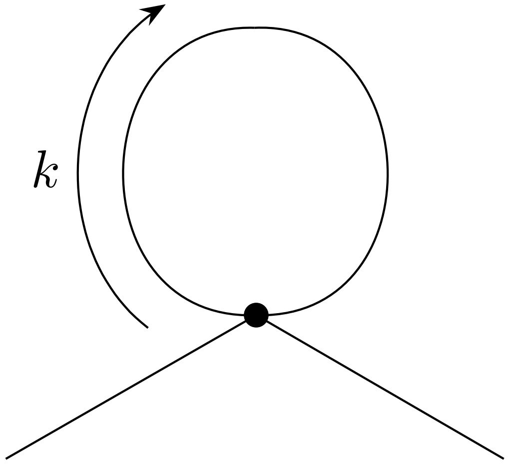
$$\sim \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{D_1 D_2 D_3 D_4}$$

$$D_1 = k^2 - m^2$$
$$D_2 = (k + p_1)^2 - m^2$$
$$D_3 = (k + p_1 + p_2)^2 - m^2$$
$$D_4 = (k + p_1 + p_2 - p_3)^2 - m^2$$



$$\sim \int \frac{\mathrm{d}^4 k_1}{(2\pi)^4} \int \frac{\mathrm{d}^4 k_2}{(2\pi)^4} \frac{1}{P_1 P_2 P_3 P_4 P_5 P_6}$$

Computing Integrals



$$\sim \int_{-\infty}^{\infty} \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{(k^2 - m^2 + i\delta)}$$

$$= \frac{-i}{(2\pi)^4} \int \mathrm{d}\Omega_3 \int_0^{\infty} \mathrm{d}r \; r^3 \frac{1}{(r^2 + m^2 - i\delta)}$$

$$= \frac{-i}{(2\pi)^4} \frac{2\pi^2}{\Gamma(2)} \int_0^{\infty} \mathrm{d}r \; r^3 \frac{1}{(r^2 + m^2 - i\delta)}$$

$$\sim \frac{-i}{(2\pi)^4} \frac{2\pi^2}{\Gamma(2)} \frac{1}{2} \left[r^2 - m^2 \ln(r^2 + m^2)\right]_0^{\infty}$$

Problem:
This integral is divergent!

nonsense

There are many ways out of this problem!

Note: If measure was $\mathrm{d}^D k$ then for $D < 2$ this integral would be finite, this observation led to **Dimensional Regularisation**

Aside: Divergence from $|k^\mu| \to \infty$, called an ultraviolet (UV) divergence

Dimensional Regularisation

Dim. Reg. is the current ``standard'' in perturbation theory.

Key Ideas:

- Treat number of space-time dimensions $(D = 4 - 2\epsilon) \in \mathbb{C}$
- Reformulate entire QFT in D dimensions (start from $\mathcal{L}$)
- Use $D < 4$ to regulate UV, use $D > 4$ to regulate infrared (IR)
- Physical observables for $D = 4$ are obtained by $D \to 4$ (analytic continuation)

't Hooft, Veltman 72

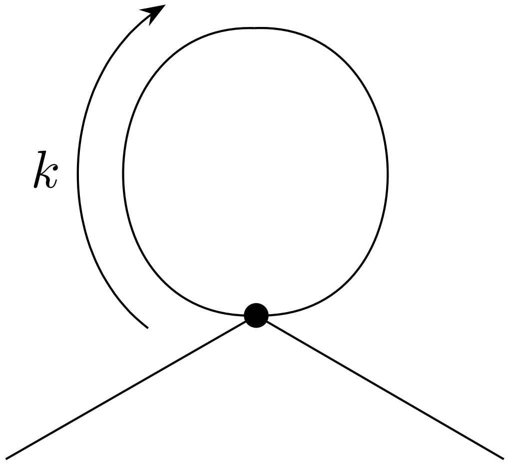
not always easy (γ_5)

For this to be consistent we require (1) **uniqueness**, (2) **existence** and we need to know (3) **properties** (linearity, scaling, translation invar.)

Recommended: J. Collins, Renormalization

⟶ **See textbook**

Computing Integrals (Revisited)



$$\sim I = \int \frac{\mathrm{d}^{D}k}{(2\pi)^D} \frac{1}{(k^2 - m^2 + i\delta)}$$

$$= \frac{-i}{(2\pi)^D} \int \mathrm{d}\Omega_{D-1} \int_0^\infty \mathrm{d}r \; r^{D-1} \frac{1}{(r^2 + m^2 - i\delta)}$$

$$= \frac{-i}{(2\pi)^D} \frac{2\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2})} \int_0^\infty \mathrm{d}r \; r^{D-1} \frac{1}{(r^2 + m^2 - i\delta)}$$

Substitute: $r^2 = y(m^2 - i\delta)$

$$I = \frac{-i}{(2\pi)^D} \frac{2\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2})} \frac{1}{2} (m^2 - i\delta)^{\frac{D}{2}-1} \int_0^\infty \mathrm{d}y \; y^{\frac{D}{2}-1} (y+1)^{-1}$$

$$= \frac{-i}{(2\pi)^D} \frac{2\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2})} \frac{1}{2} (m^2 - i\delta)^{\frac{D}{2}-1} \mathrm{B}\left(\frac{D}{2}, 1 - \frac{D}{2}\right)$$

$$= \frac{-i}{(2\pi)^D} \frac{2\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2})} \frac{1}{2} (m^2 - i\delta)^{\frac{D}{2}-1} \frac{\Gamma(\frac{D}{2})\Gamma(1 - \frac{D}{2})}{\Gamma(1)}$$

Euler Beta Function

$\mathrm{Re}(D/2) > 0$

$\mathrm{Re}(1 - D/2) > 0$

Our problem is solved! How do we do more complicated integrals?

Part 2

There are many ways of computing Feynman integrals!

What follows is one specific approach.

Conventions

Loop integral:

L loops

$$G = \int \prod_{l=1}^{L} [\mathrm{d}^D k_l] \, \frac{1}{\prod_{j=1}^{N} P_j^{\nu_j}(\{k\},\{p\},m_j^2)}$$

N propagators

$$[\mathrm{d}^D k_l] = \frac{\mu^{4-D}}{i\pi^{\frac{D}{2}}}\mathrm{d}^D k_l$$

Propagator:

Mass

Important: +

$$P_j(\{k\},\{p\},m_j^2) = (q_j^2 - m_j^2 + i\delta)$$

Loop momenta

External momenta

Linear combination of loop/ external momenta

Feynman Parameterization

Previous integral was easy due to **spherical symmetry**! Feynman parameterization is one way to cast all loop integrals into this form.

Notice that:

$$\frac{1}{AB} = \int_0^1 \frac{du}{\left[uA + (1-u)B\right]^2}$$

Or more generally:

$$\frac{1}{\prod_{j=1}^{N} P_j^{\nu_j}} = \frac{\Gamma(N_\nu)}{\prod_{j=1}^{N} \Gamma(\nu_j)} \int_0^\infty \prod_{j=1}^{N} \mathrm{d}x_j x_j^{\nu_j - 1} \delta(1 - \sum_{i=1}^{N} x_i) \frac{1}{\left[\sum_{j=1}^{N} x_j P_j\right]^{N_\nu}}$$

Product

Feynman Parameters

Sum

$$N_\nu = \nu_1 + \ldots + \nu_N$$

Feynman Parameterization (II)

Feynman parameterizing our loop integral:

$$G = \int_{-\infty}^{\infty} \prod_{l=1}^{L} [\mathrm{d}^D k_l] \, \frac{1}{\prod_{j=1}^{N} P_j^{\nu_j}} = \frac{\Gamma(N_\nu)}{\prod_{j=1}^{N} \Gamma(\nu_j)} \int_0^\infty \prod_{j=1}^{N} \mathrm{d}x_j x_j^{\nu_j - 1} \delta(1 - \sum_{i=1}^{N} x_i)$$

$$\times \int_{-\infty}^{\infty} \prod_{l=1}^{L} [\mathrm{d}^D k_l] \left[\sum_{i,j=1}^{L} k_i^T M_{ij} k_j - 2 \sum_{j=1}^{L} k_j^T \cdot Q_j + J + i\delta \right]^{-N_\nu}$$

From quadratic (in k) terms of propagators

Linear (in k) terms

Key Point: In this form we can shift k to eliminate linear terms (obtain spherical symmetry) then do the momentum integrals!

Feynman Parameterization (III)

After integration over momenta we obtain:

Master Formula

$$G = (-1)^{N_\nu} \frac{\Gamma(N_\nu - LD/2)}{\prod_{j=1}^{N} \Gamma(\nu_j)} \int_0^\infty \prod_{j=1}^{N} \mathrm{d}x_j \; x_j^{\nu_j - 1} \delta(1 - \sum_{i=1}^{N} x_i) \frac{\mathcal{U}^{N_\nu - (L+1)D/2}(\vec{x})}{\mathcal{F}^{N_\nu - LD/2}(\vec{x}, s_{ij})}$$

Graph Polynomials:

1st Symanzik Polynomial: $\mathcal{U}(\vec{x}) = \det(M)$

2nd Symanzik Polynomial: $\mathcal{F}(\vec{x}, s_{ij}) = \det(M) \left[\sum_{i,j=1}^{L} Q_i M_{ij}^{-1} Q_j - J - i\delta \right]$

We have exchanged L momentum integrals for N parameter integrals

Maybe this looks complicated… but wait!

Graph Polynomials

Properties:

- Homogenous polynomials in the Feynman Parameters

  $\mathcal{U}(\vec{x})$ is degree L

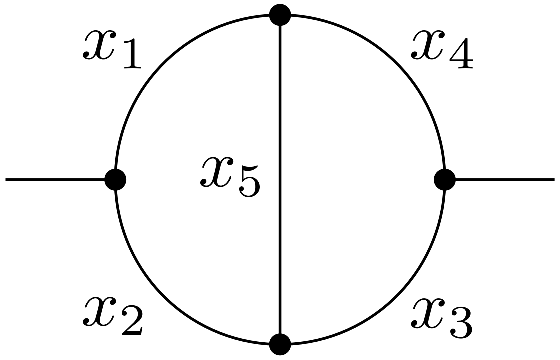
  $\mathcal{F}(\vec{x}, s_{ij})$ is degree $L+1$

  $\mathcal{F}(\vec{x}, s_{ij}) = \mathcal{F}_0(\vec{x}, s_{ij}) + \mathcal{U}(\vec{x}) \sum_{i=1}^{N} x_i m_i^2$ ← **Internal masses**
- $\mathcal{U}(\vec{x})$ and $\mathcal{F}_0(\vec{x}, s_{ij})$ are linear in each Feynman Parameter

$\mathcal{F}_0(\vec{x}, s_{ij})$ **and** $\mathcal{U}(\vec{x})$ **can be constructed graphically**

We will follow: Bogner, Weinzierl 10

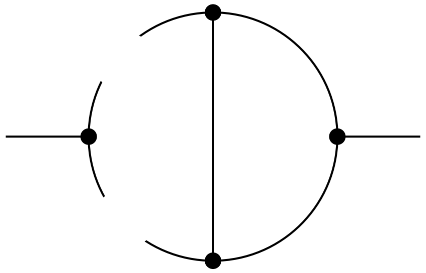
Constructing U



Draw graph, label edges with Feynman Parameters

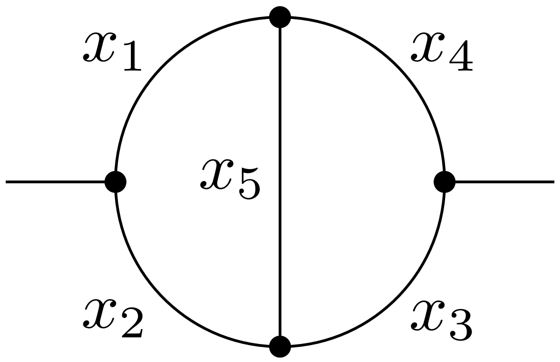
Rules for $\mathcal{U}(\vec{x})$**:**

1. Delete L edges all possible ways
2. Throw away disconnected graphs or graphs with $L \neq 0$
3. Sum monomials of Feynman parameters of deleted edges



$\mathcal{U}(\vec{x}) =$

Constructing U



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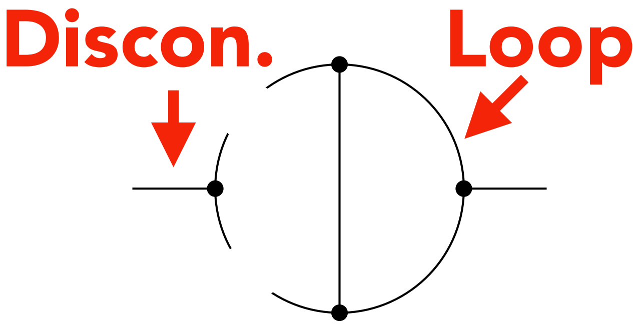

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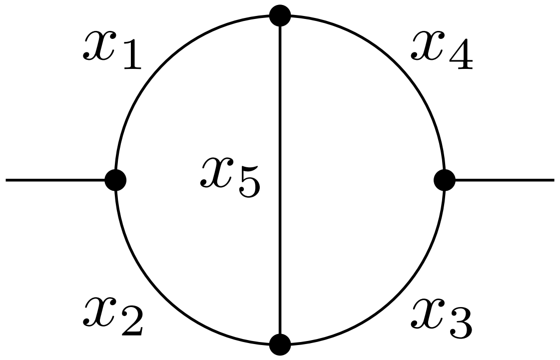

Draw graph, label edges with Feynman Parameters

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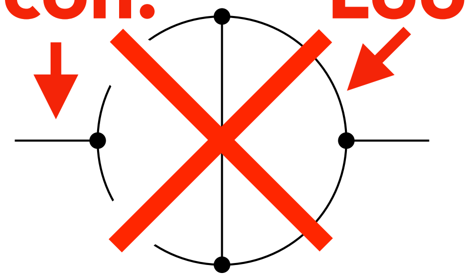
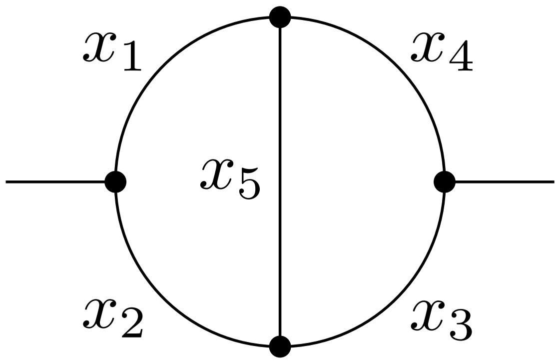
Discon. **Loop**



$\mathcal{U}(\vec{x}) =$

Constructing U



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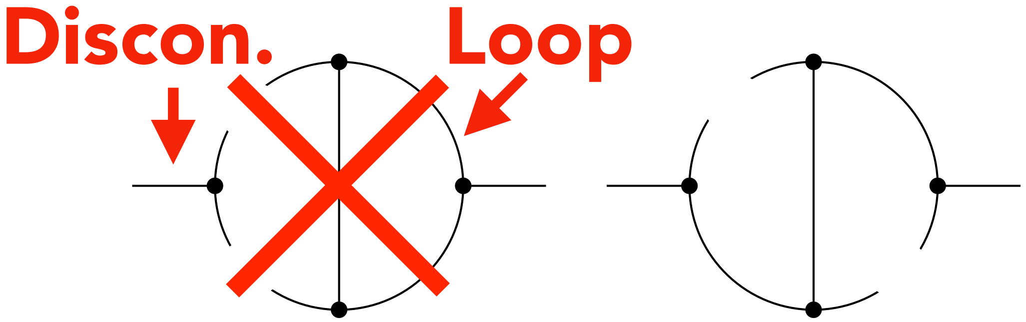
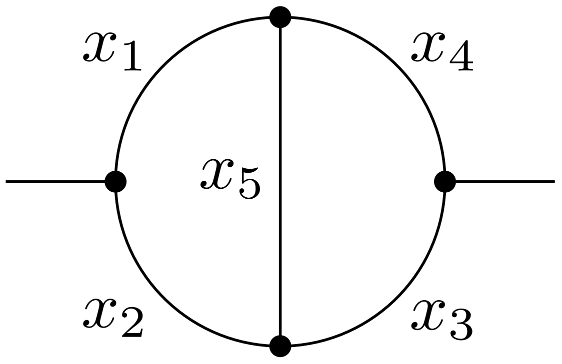


$\mathcal{U}(\vec{x}) =$ $+x_1 x_3$

Constructing U

Draw graph, label edges with Feynman Parameters



Rules for $\mathcal{U}(\vec{x})$:

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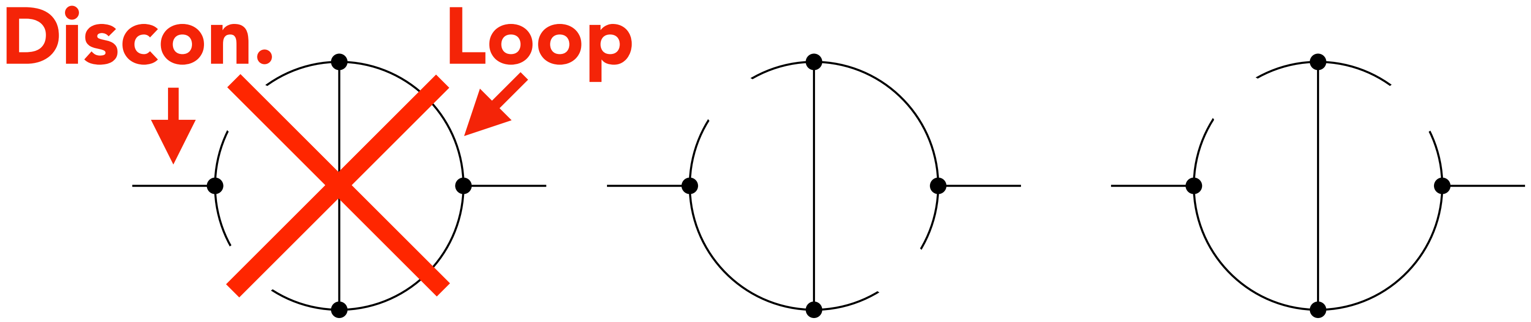
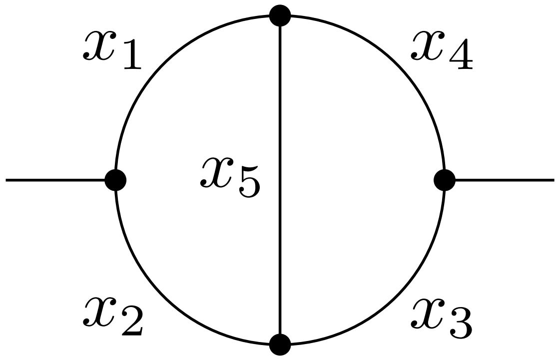


$\mathcal{U}(\vec{x}) = \qquad + x_1 x_3 \qquad + x_1 x_4$

Constructing U



Draw graph, label edges with Feynman Parameters

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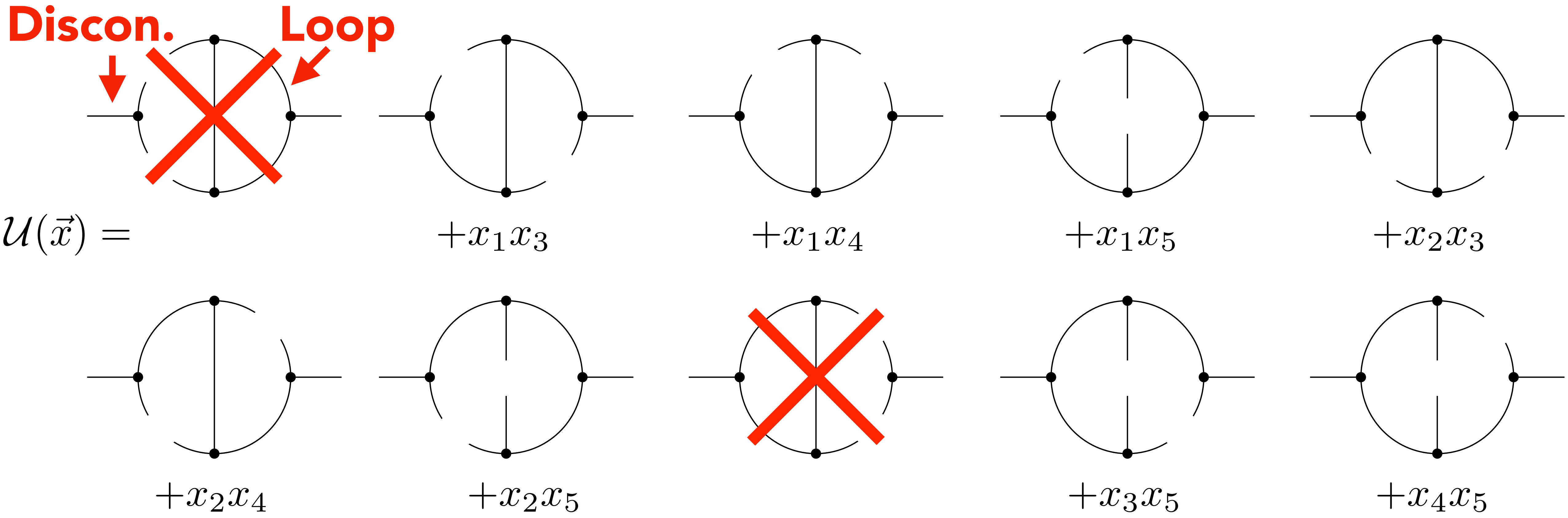


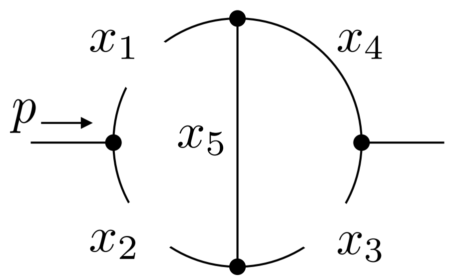
$$\mathcal{U}(\vec{x}) = +x_1x_3 +x_1x_4 +x_1x_5 +x_2x_3 +x_2x_4 +x_2x_5 +x_3x_5 +x_4x_5$$

Constructing F

Rules for $\mathcal{F}_0(\vec{x}, s_{ij})$:

1. Delete $L+1$ edges all possible ways
2. Take only graphs with 2 connected components (T1, T2) and $L=0$
3. Sum F.P. monomials multiplied by: $-s_{ij} = -(\sum_k q_k)^2$
4. (For $\mathcal{F}(\vec{x}, s_{ij})$ add the internal mass terms)

Momenta flowing through cut lines from T1 → T2



$$\mathcal{F}_0 = \ -p^2 x_1 x_2 x_3$$

Constructing F

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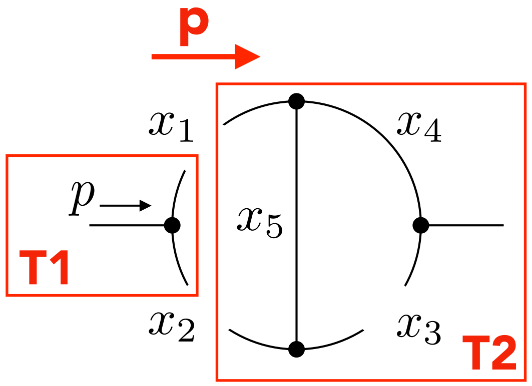
4. (For $\mathcal{F}(\vec{x}, s_{ij})$ add the internal mass terms)

Momenta flowing through cut lines from T1 → T2



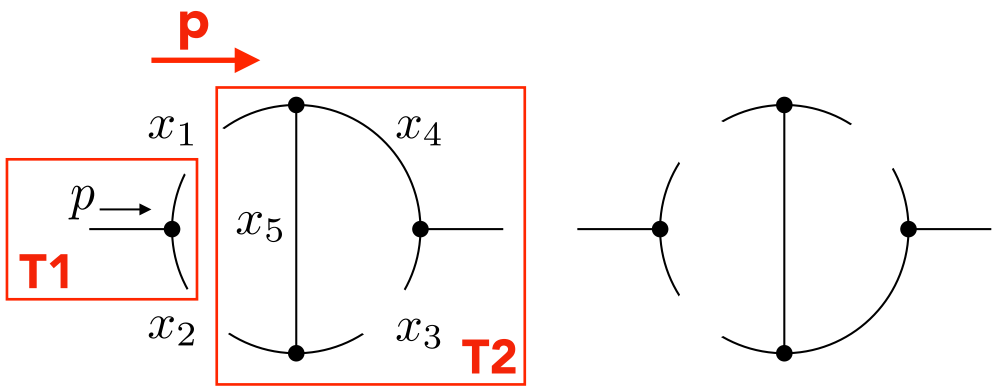
$$\mathcal{F}_0 = \; -p^2 x_1 x_2 x_3$$

Constructing F

Rules for $\mathcal{F}_0(\vec{x}, s_{ij})$ **:**

1. Delete $L+1$ edges all possible ways
2. Take only graphs with 2 connected components (T1, T2) and $L = 0$
3. Sum F.P. monomials multiplied by: $-s_{ij} = -(\sum_k q_k)^2$
4. (For $\mathcal{F}(\vec{x}, s_{ij})$ add the internal mass terms)

Momenta flowing through cut lines from T1 → T2



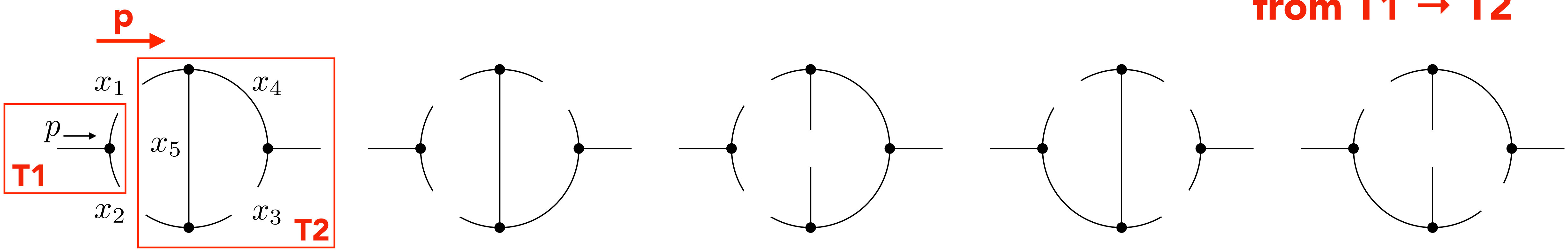
$$\mathcal{F}_0 = \quad -p^2 x_1 x_2 x_3 \qquad\qquad -p^2 x_1 x_2 x_4$$

Constructing F

Rules for $\mathcal{F}_0(\vec{x}, s_{ij})$:

1. Delete $L+1$ edges all possible ways
2. Take only graphs with 2 connected components (T1, T2) and $L = 0$
3. Sum F.P. monomials multiplied by: $-s_{ij} = -(\sum_k q_k)^2$
4. (For $\mathcal{F}(\vec{x}, s_{ij})$ add the internal mass terms)

Momenta flowing through cut lines from T1 → T2



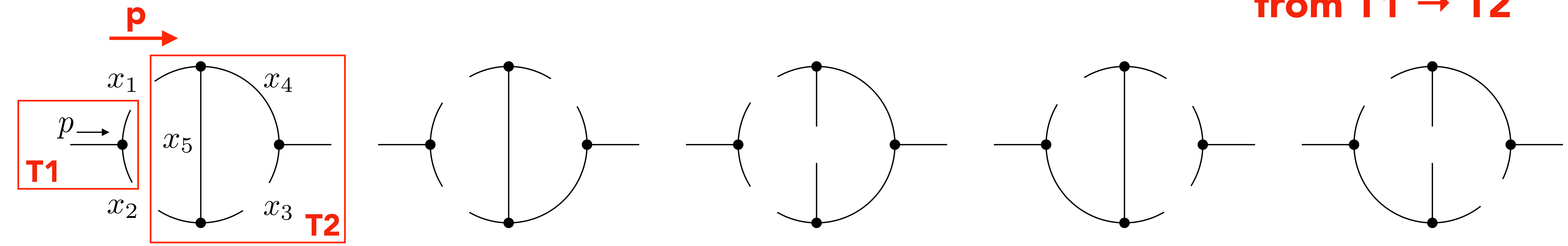
$$\mathcal{F}_0 = \quad -p^2 x_1 x_2 x_3 \quad -p^2 x_1 x_2 x_4 \quad -p^2 x_1 x_2 x_5 \quad -p^2 x_1 x_3 x_4 \quad -p^2 x_1 x_3 x_5$$

Constructing F

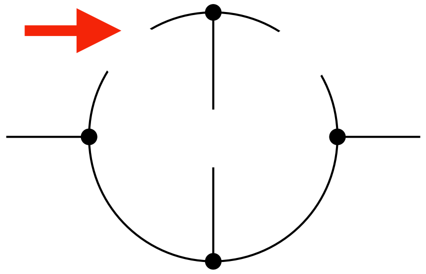
Rules for $\mathcal{F}_0(\vec{x}, s_{ij})$ **:**

1. Delete $L+1$ edges all possible ways
2. Take only graphs with 2 connected components (T1, T2) and $L=0$
3. Sum F.P. monomials multiplied by: $-s_{ij} = -(\sum_k q_k)^2$
4. (For $\mathcal{F}(\vec{x}, s_{ij})$ add the internal mass terms)

Momenta flowing through cut lines from T1 → T2



$$\mathcal{F}_0 = \quad -p^2x_1x_2x_3 \qquad -p^2x_1x_2x_4 \qquad -p^2x_1x_2x_5 \qquad -p^2x_1x_3x_4 \qquad -p^2x_1x_3x_5$$



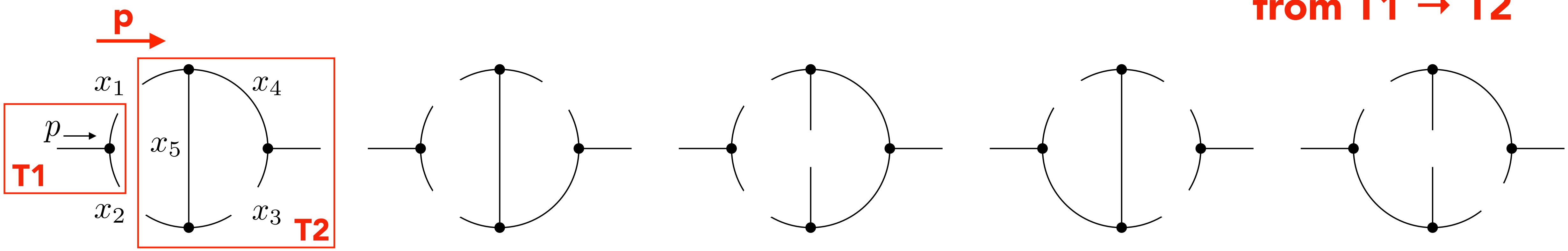
$$-0^2x_1x_4x_5$$

Constructing F

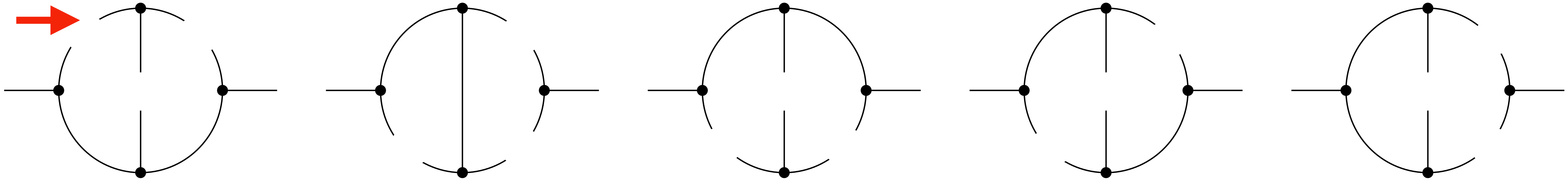
Rules for $\mathcal{F}_0(\vec{x}, s_{ij})$ **:**

1. Delete $L+1$ edges all possible ways
2. Take only graphs with 2 connected components (T1, T2) and $L=0$
3. Sum F.P. monomials multiplied by: $-s_{ij} = -(\sum_k q_k)^2$
4. (For $\mathcal{F}(\vec{x}, s_{ij})$ add the internal mass terms)

Momenta flowing through cut lines from T1 → T2



$$\mathcal{F}_0 = -p^2 x_1 x_2 x_3 \quad -p^2 x_1 x_2 x_4 \quad -p^2 x_1 x_2 x_5 \quad -p^2 x_1 x_3 x_4 \quad -p^2 x_1 x_3 x_5$$



$$-0^2 x_1 x_4 x_5 \quad -p^2 x_2 x_3 x_4 \quad -0^2 x_2 x_3 x_5 \quad -p^2 x_2 x_4 x_5 \quad -p^2 x_3 x_4 x_5$$

Divergences

From the master formula, 3 possibilities for poles in ϵ to arise:

1. Overall $\Gamma(N_\nu - LD/2)$ diverges (single UV pole)
2. $\mathcal{U}(\vec{x})$ vanishes for some $x = 0$ and has negative exponent (UV sub-divergences)
3. $\mathcal{F}(\vec{x}, s_{ij})$ vanishes on the boundary and has negative exponent (IR divergences)

Outside the Euclidean region ($\forall s_{ij} < 0$) there is a further possibility:

4. $\mathcal{F}(\vec{x}, s_{ij})$ vanishes inside the integration region (May give: Landau singularity which is either a normal or anomalous threshold)

Not discussed here (can be handled by `SecDec:` contourdef=True)

See: Soper 00; Borowka 14

Aside: If only condition 1 leads to a divergence the integral is **Quasi-finite**

Sector Decomposition

We are now faced with integrals of the form:

$$G_i = \int_0^1 \left(\prod_{j=1}^{N-1} \mathrm{d}x_j x_j^{\nu_j - 1} \right) \frac{\mathcal{U}_i(\vec{x})^{\mathrm{expo}\mathcal{U}(\epsilon)}}{\mathcal{F}_i(\vec{x}, s_{ij})^{\mathrm{expo}\mathcal{F}(\epsilon)}}$$

Powers depending on ϵ

Polynomials in F.P

Which may contain **overlapping singularities** which appear when several $x_j \to 0$ simultaneously

Sector decomposition maps each integral into integrals of the form:

$$G_{ik} = \int_0^1 \left(\prod_{j=1}^{N-1} \mathrm{d}x_j x_j^{a_j - b_j \epsilon} \right) \frac{\mathcal{U}_{ik}(\vec{x})^{\mathrm{expo}\mathcal{U}(\epsilon)}}{\mathcal{F}_{ik}(\vec{x}, s_{ij})^{\mathrm{expo}\mathcal{F}(\epsilon)}}$$

Singularity structure can be read off

$$\mathcal{U}_{ik}(\vec{x}) = 1 + u(\vec{x})$$

$$\mathcal{F}_{ik}(\vec{x}) = -s_0 + f(\vec{x})$$

$u(\vec{x}), f(\vec{x})$ **have no constant term**

Hepp 66; Denner, Roth 96; Binoth, Heinrich 00

Sector Decomposition (II)

One technique **Iterated Sector Decomposition** repeat:

Binoth, Heinrich 00

$$\int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2 \frac{1}{(x_1+x_2)^{2+\epsilon}}$$

← **Overlapping singularity for** $x_1, x_2 \to 0$

$$= \int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2 \frac{1}{(x_1+x_2)^{2+\epsilon}} \left(\theta(x_1 - x_2) + \theta(x_2 - x_1)\right)$$

$$= \int_0^1 \mathrm{d}x_1 \int_0^{x_1} \mathrm{d}x_2 \frac{1}{(x_1+x_2)^{2+\epsilon}} + \int_0^1 \mathrm{d}x_2 \int_0^{x_2} \mathrm{d}x_1 \frac{1}{(x_1+x_2)^{2+\epsilon}}$$

$$= \int_0^1 \mathrm{d}x_1 \int_0^{1} \mathrm{d}t_2 \frac{x_1}{(x_1+x_1 t_2)^{2+\epsilon}} + \int_0^1 \mathrm{d}x_2 \int_0^{1} \mathrm{d}t_1 \frac{x_2}{(x_2 t_1+x_2)^{2+\epsilon}}$$

$$= \int_0^1 \mathrm{d}x_1 \int_0^{1} \mathrm{d}t_2 \frac{x_1^{-1-\epsilon}}{(1+t_2)^{2+\epsilon}} + \int_0^1 \mathrm{d}x_2 \int_0^{1} \mathrm{d}t_1 \frac{x_2^{-1-\epsilon}}{(t_1+1)^{2+\epsilon}}$$

← **Singularities factorised**

If this procedure terminates depends on order of decomposition steps

An alternative strategy **Geometric Sector Decomposition** always terminates; both strategies are implemented in `SecDec`.

Kaneko, Ueda 10; See also: Bogner, Weinzierl 08; Smirnov, Tentyukov 09

Extraction

Consider a **Sector Decomposed** integral (simple case $a = -1$):

$$\int_0^1 \mathrm{d}x x^{-1-b\epsilon} f(x)$$

$$= \int_0^1 \mathrm{d}x x^{-1-b\epsilon} \left[f(x) - f(0) + f(0)\right]$$

$$= \int_0^1 \mathrm{d}x x^{-1-b\epsilon} f(0) + \int_0^1 \mathrm{d}x x^{-1-b\epsilon} \left[f(x) - f(0)\right]$$

$$= \frac{f(0)}{-b\epsilon} + \int_0^1 \mathrm{d}x x^{-b\epsilon} \left[\frac{f(x) - f(0)}{x}\right]$$

← **Finite**

Poles

By Definition:

$f(0) \neq 0$

$f(0)$ finite

Key Point: Sector Decomposed integrals can be easily expanded in ϵ and **numerically** integrated!

Part 3

Demo

Warning

1. F.P representation can sometimes obscure properties of integrals, can calculate the 2-loop propagator type integral to all orders in ϵ analytically but this was not obvious from the F.P representation
2. Sector Decomposition itself can make the analytical structure of integrals more complicated (by introducing spurious transcendental functions)

von Manteuffel, Schabinger, Zhu 12; von Manteuffel, Panzer, Schabinger 14

Last but not least:

3. `SecDec` integrates functions numerically - this can be slow.

But: can compute complicated (unknown) multi-scale integrals automatically often with reasonable wall time & provides an automated cross-check for other methods

`SecDec`

`SecDec` (https://secdec.hepforge.org)

Evaluate Dimensionally regulated parameter integrals numerically

Collaboration: Borowka, Heinrich, Jahn, SJ, Kerner, Schlenk, Zirke

Many examples in directory: `loop/demos`

Supports (within reason):

- Arbitrary Loops & Legs
- Numerators, Inverse propagators, ``Dots”
- Euclidean & Physical Kinematics
- Linear Propagators
- Arbitrary (Complex) Masses/ Off-shellness
- … (General parameter integrals, see: `general/demos`)

I did not speak about this

Also…

Other public programs which implement Sector Decomposition:

`FIESTA`
`(http://science.sander.su/FIESTA.htm)`

Smirnov, Tentyukov

`sector_decomposition + CSectors`
`(http://wwwthep.physik.uni-mainz.de/~stefanw/sector_decomposition)`

Bogner, Weinzierl; Gluza, Kajda, Riemann, Yundin

Installation

Dependencies:

- `Mathematica` 7+
- Perl, C++ Compiler
- (Optional) For Geometric Decomposition: `Normaliz`
- (Included) `Cuba`, `Bases`, `CQUAD`

Bruns, Ichim, Roemer, Soeger

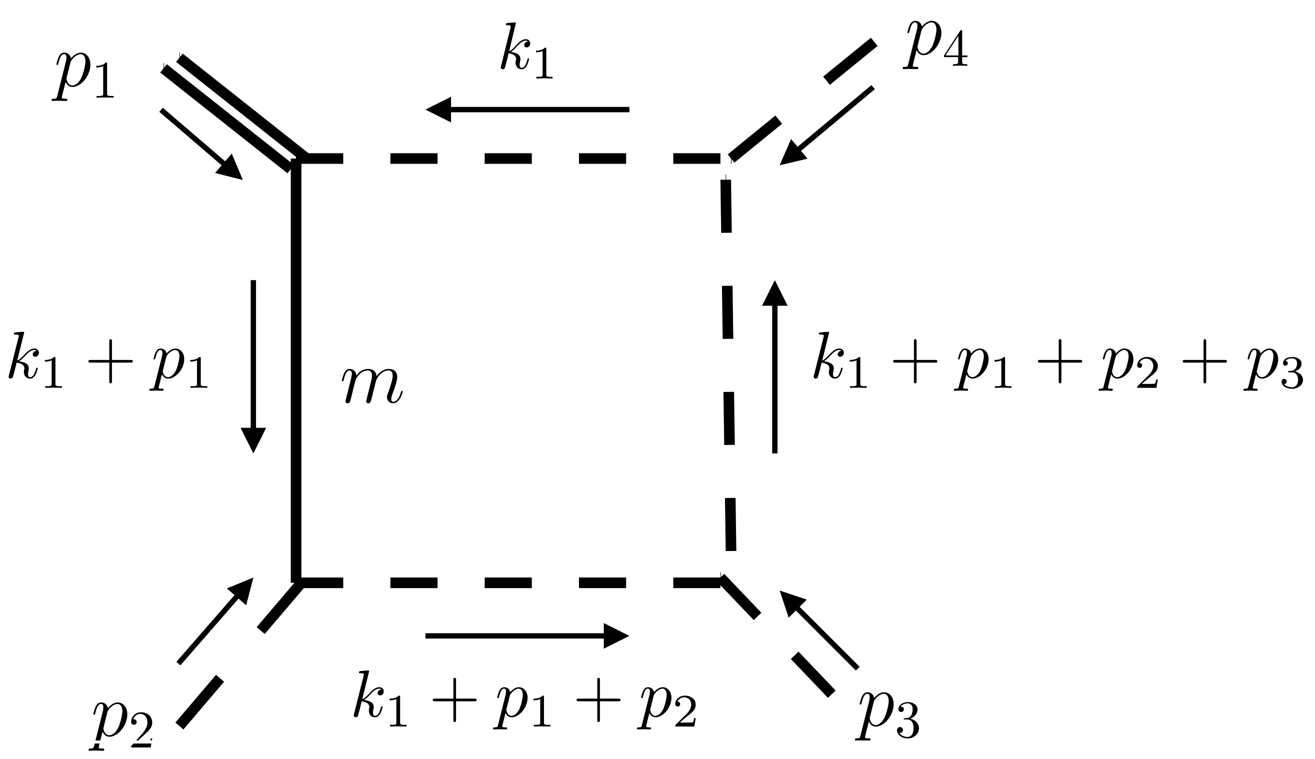
Hahn; Kawabata; Gonnet

Installation:

```
tar -xzvf SecDec-3.0.8.tar.gz
cd SecDec-3.0.8
make
(make check)
```

Example 1: box_1L

1-loop Box



Propagators:

$$(k_1)^2$$
$$(k_1 + p_1)^2 - m^2$$
$$(k_1 + p_1 + p_2)^2$$
$$(k_1 + p_1 + p_2 + p_3)^2$$

Scalar Products:

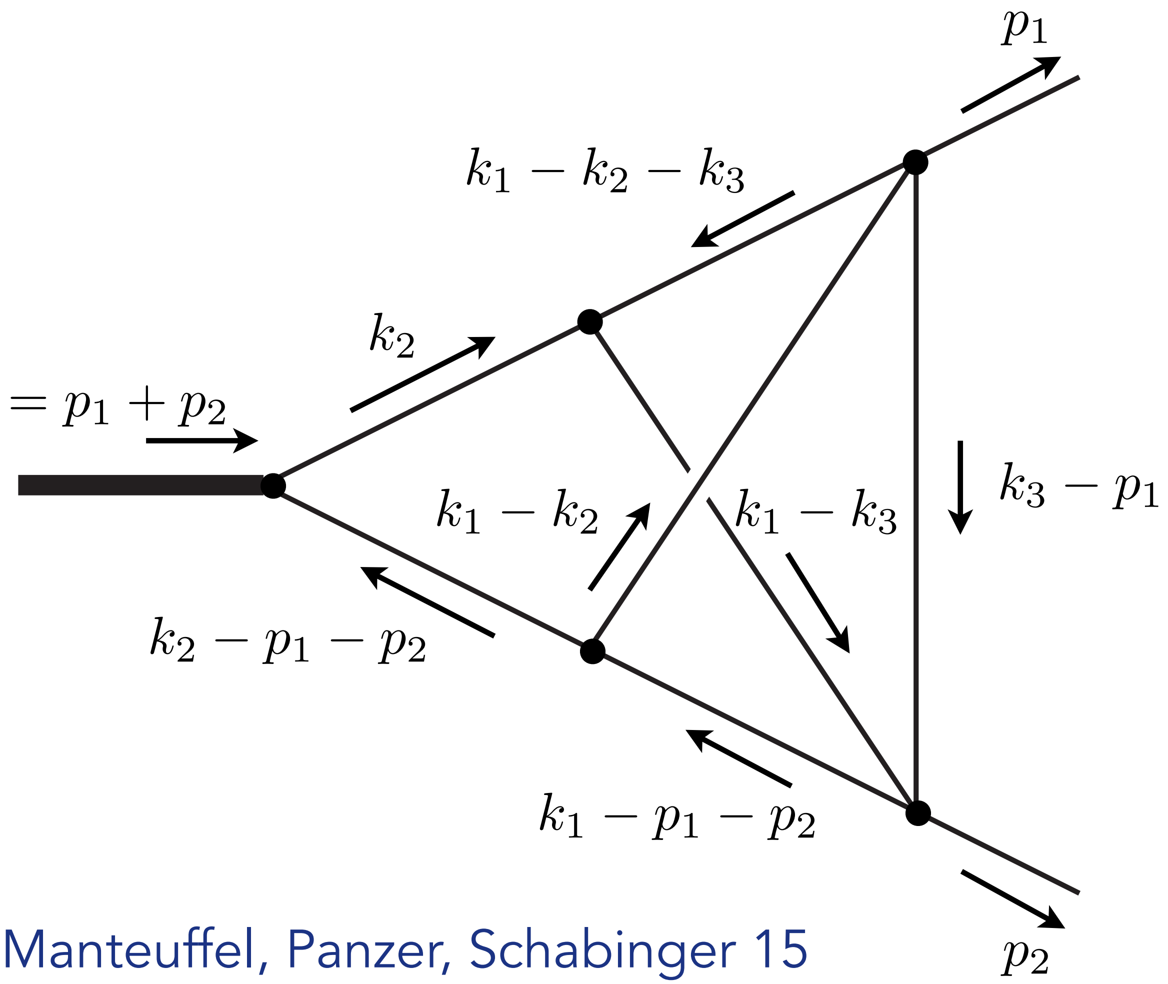
$$p_1 \cdot p_1 = s_1$$
$$p_1 \cdot p_2 = s/2 - s_1/2$$
$$p_1 \cdot p_3 = -t/2 - s/2$$
$$p_1 \cdot p_4 = t/2 - s_1/2$$

$$p_2 \cdot p_2 = 0$$
$$p_2 \cdot p_3 = t/2$$
$$p_2 \cdot p_4 = s_1/2 - t/2 - s/2$$

$$p_3 \cdot p_3 = 0$$
$$p_3 \cdot p_4 = s/2$$
$$p_4 \cdot p_4 = 0$$

Example 2: ff_3L

Massless 3-loop Form Factor



von Manteuffel, Panzer, Schabinger 15

Propagators:

$$(k_2)^2$$
$$(k_1 - k_2)^2$$
$$(k_1 - k_3)^2$$
$$(k_1 - k_2 - k_3)^2$$
$$(k_1 - p_1 - p_2)^2$$
$$(k_2 - p_1 - p_2)^2$$
$$(k_3 - p_1)^2$$

Scalar Products:

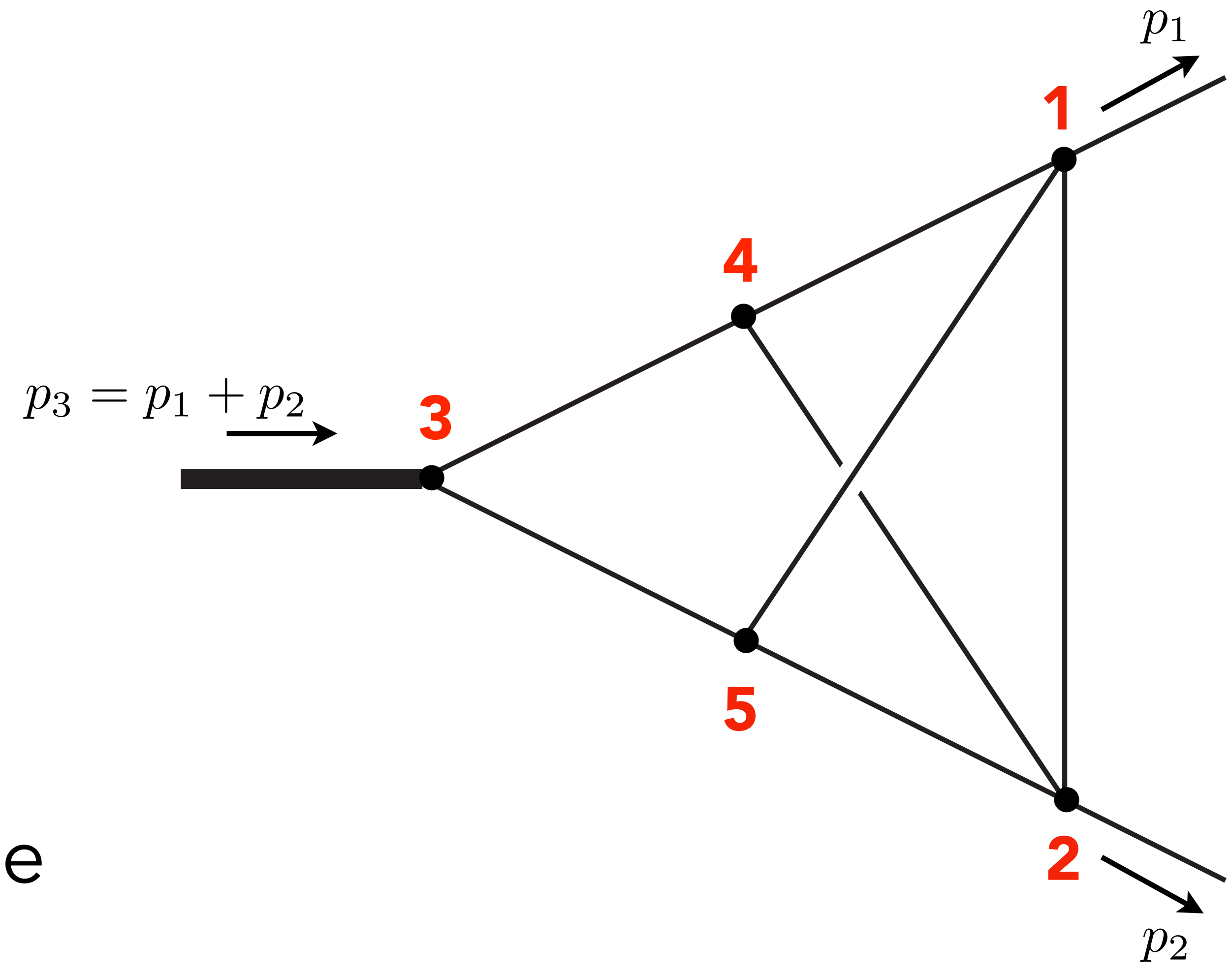
$$p_1 \cdot p_1 = 0 \qquad p_2 \cdot p_2 = 0 \qquad p_3 \cdot p_3 = s$$
$$p_1 \cdot p_2 = s/2 \qquad p_2 \cdot p_3 = -s/2 \qquad p_1 \cdot p_3 = -s/2$$

Example 2: ff_3L (II)

Alternatively, rather than **propagators** we can specify an **adjacency list**

Note: Vertices connected to external momenta must be numbered correctly!

ExternalMomenta = {p1,p2,p3};
Position: 1 2 3



Mass of edge

Adjacency List:

{ {0,{1,2}}, {0,{1,4}}, {0,{1,5}}, {0,{2,4}}, {0,{2,5}}, {0,{3,4}}, {0,{3,5}} }

Aside: Integral is finite, technically do not **need** Sector Decomposition

Thank you for listening!